MAA Reviews

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Rational Algebraic Curves: A Computer Algebra Approach

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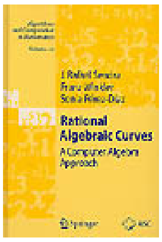

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MAA Review

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Algebraic geometry is a central field within mathematics which is often viewed as difficult for outsiders to enter. Recent years have seen many books which present various topics of algebraic geometry from a computational viewpoint, thereby making the subject more accessible. The book under review, *RAC* for short, is part of this positive trend. Its main focus is parametrizing rational plane algebraic curves. It is a graduate level text aimed at fairly wide readership, including for example readers interested primarily in computer-aided geometric design.

To give an idea of the content and level of *RAC*, I will slowly work out its Exercise 4.20:

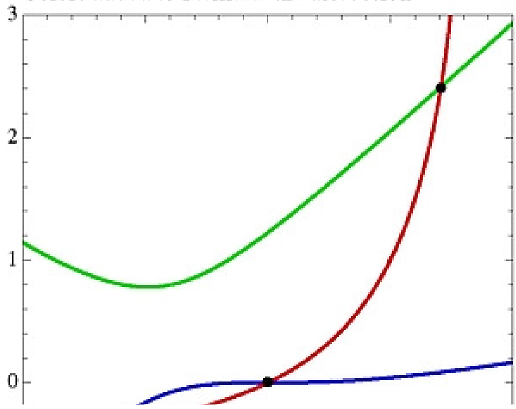
Let *C* be the affine curve defined by

$$f(x,y) = x^4 - 11xy - 9x^2y - 6x^3y + 23y^2 + 16xy^2 - 20y^3 + 5xy^3 + y^4 = 0.$$

Compute a rational parametrization of *C* .

The problem has been posed in purely algebraic terms and, as we'll see, can be solved in purely algebraic terms. But ignoring geometry would be missing half of algebraic geometry! So we type the given polynomial into a computer algebra system to draw part of *C* in the x-y plane. We thereby obtain a monochromatic version of Figure 1. We learn immediately that at three points in this window, the curve *C* looks locally like the letter *X*. The three crossing points are called singularities, and they will play an important role in our algebraic solution. Figure 1 draws these points as small black disks.

Figure 1. Part of *C*, drawn in the usual x–y plane. Colors added to facilitate the discussion.

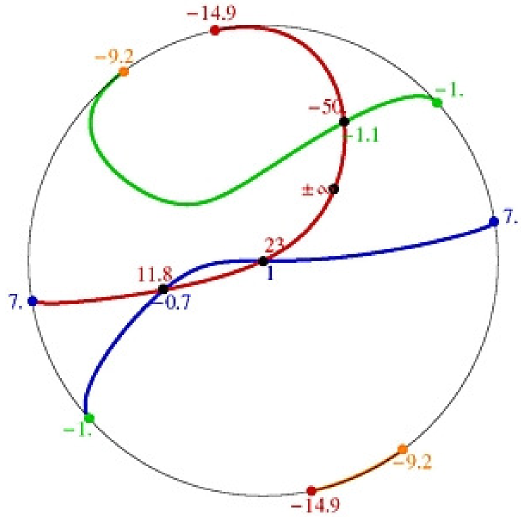


Continuing with geometry, a problem with Figure 1 is that it shows only part of *C*. To remedy this problem, we can use the coordinates $(u,v) = (x,y)/(r^2 + x^2 + y^2)^{1/2}$ for any positive real r we find convenient. Via these coordinates, we can draw the entire x-y plane as the open unit disk U in the u-v plane. Exactly half the area of U comes from the disk of radius r about the origin in the x-y plane. Accordingly, one can take r small or large according to whether one wants to dedicate visual space to the part of the x-y plane near or far from the origin. In Figure 2, we view all of *C* in this way, taking r=3. So we learn that Figure 1 missed a whole piece of our *C*, the part drawn in orange at the lower right of Figure 2. In fact, Figure 1 missed this piece by a lot, since the closest the orange part of *C* comes to the origin is at $(x,y) \approx (20,-45)$.

Figure 2 has actually given us more insight than we asked of it, because the circle ∂U bounding U also plays a role. In fact, if we identify opposite points of this boundary circle then we have just abstractly sewn the closed unit disk $\overline{U}$ into the "projective plane." Our affine curve *C* has gained four new points to become its projective completion $\overline{C}$.

Now let's mix algebraic and geometric thinking, as even beginning algebraic geometers should! In elementary algebraic terms, Exercise 4.20 is asking for non-constant rational functions x(t) and y(t) of minimal degree such that f(x(t),y(t)) is identically zero. Geometrically, one can think of t as time, and then (x(t),y(t)) should be thought of as a moving point. The numbers near the curve on Figure 2 capture the solution we will be algebraically producing. At $t= -\infty$, the point starts at the point (1,1) which is labeled $\pm\infty$ on Figure 2. Then, as t increases, the point (x(t),y(t)) moves first upwards on the red arc. It goes straight through points at infinity, and also straight through singular points, moving mostly from left to right. Finally at $t=\infty$ the point returns to (1,1), having visited each singular point twice and all other points exactly once. Thus — despite our color scheme! — the curve $\overline{C}$ forms a single loop.

Figure 2. All of $\overline{C}$, drawn in the projective plane. The numbers are t−values of the parametrization.



If it shocks your mathematical intuition that a plane curve *C* given by a random f(x,y) should be so parametrizable, then you are right! Only very special curves are parametrizable. If you are intimidated also about passing from abstract existence to actually finding (x(t),y(t)), then you are again reacting properly. In practice, it would be impossible to find (x(t),y(t)) by naive algebraic fiddling with variables. A systematic geometry-inspired approach is required, and that is the subject of *RAC*!

To do at least some justice to the systematic approach of *RAC*, let's consider a general polynomial f(x,y) with real coefficients. Let d be its degree, i.e. the largest i+j appearing among its terms ax^iy^j. Then a sufficient condition for parametrizability is that the corresponding complete curve $\overline{C}$ consists of a single loop which crosses itself at $(d-1)(d-2)/2$ singular points in the x-y plane. This sufficient condition is satisfied in our case, since d=4 and $(4-1)(4-2)/2=3$. A weaker but similar condition, involving complex numbers among other things, is necessary and sufficient for parametrizability.

One of *RAC*'s central algorithms is parametrization-by-adjoints in Section 4.7. Actually using this algorithm to parametrize a curve meeting our sufficient condition is a very attractive mix of algebra and geometry. The algorithm comes in several variants, as the adjoint curves involved can have degree d−2, d−1 or d. We will use the d−2 variant for our *C*, simultaneously indicating how it works for general $d \geq 3$.

First, one locates the singularities of *C* by finding the common roots of f(x,y) and its partial derivatives $f_x(x,y)$ and $f_y(x,y)$. This is a standard computer algebra task. In our case, the three singularities are (0,0), $(-\sqrt{2},1-\sqrt{2})$, and $(\sqrt{2},1+\sqrt{2})$. Second, one chooses d−3 non-singular points of *C*. In our case, we need just one point and we choose (1,1). Now let *V* be the vector space of polynomials of degree $\leq d-2$. In our case, the general element of *V* has the form

$$g(x,y) = a\,x^2 + b\,x\,y + c\,y^2 + d\,x + e\,y + f.$$

Let *A* be the subspace of *V* consisting of those g(x,y) which vanish on all (d−1)(d−2)/2 singular points and also on the d−3 chosen non-singular points. A key feature of this construction, expected by a naïve dimension count, is that *A* always has dimension two. Let $g_0(x,y)$ and $g_1(x,y)$ be a basis for *A*, and form the one parameter family of polynomials $g_t(x,y) = (1-t)\, g_0(x,y) + t\, g_1(x,y)$. In our case, suitable choices yield

$$g_t(x,y) = 2x^2 + (-5 - t)\, x y + 2 y^2 + (1 - t)\, x + 2 t y.$$

Let D_t be the solution curve of $g_t(x,y)=0$. The D_t are the adjoint curves in question. Another key feature of this construction, a consequence of Bezout's theorem this time, is that for all but finitely many t, the curves *C* and D_t meet at exactly one point beyond the imposed intersection points. The remaining intersection point (x(t),y(t)), unlike the imposed ones, varies with t. To find x(t), one eliminates y from the system

$$f(x,y) = g_t(x,y)=0$$

and solves for x in terms of t. This is standard computer algebra, essentially a single call to a resultant command. Likewise, to find y(t) one eliminates x and solves for y. In our case, the final answer is

$$x(t) = (t-1)(t-23)(t^2 + 20t + 23)/c(t)$$

$$y(t) = (t-1)^3 (t-23)/c(t)$$

with $c(t) = t^4 + 18t^3 - 16t^2 - 994t - 945$. We are finally done with Exercise 4.20!

As *RAC* rightly emphasizes, a parametrization for a curve gives one much better control over the curve than one has from a defining equation alone. For example, finding points (x,y) on our *C* with rational coordinates is difficult from the original description f(x,y)=0. It is trivial from the parametrization, as one can simply plug in rational t. Likewise, the colors in Figures 1 and 2 were easy to draw only after we had the parametrization. The roots of c(t) are at approximately t=−14.9, −9.2, −1.0, and 7.0; these roots serve as start and end times for our color intervals.

In some ways, *RAC* has a classical feel. For example, as the authors indicate, the main idea of the parametrization-by-adjoints algorithm is already present in book form in Walker's classic 1950 text. In fact, Theorem III.5.1 there gives the d−1 version. However *RAC* is very modern in its emphasis on computational issues. For example, the issue of solving problems like Exercise 4.20 without ever writing down computer-unfriendly irrationalities like √2 is thoroughly treated.

I am looking forward to a future when algebraic geometry has thoroughly lost its aura of inaccessibility. Books like *RAC* are hastening the day. If you understood this review, you are ready to read *RAC*. If you were annoyed at how I suppressed complex numbers, you are more than ready!

References:

Algebraic Curves , by Robert J. Walker. Dover Publications, 1950.

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