An Algorithm for Resolution of Singularities

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Abstract

Our algorithm, based on O. Villamayor's constructive proof of existence of resolution of singularities of hypersurfaces over fields of characteristic zero, computes an embedded resolution of the singularities of a given hypersurface in affine n-space.

The currently available papers on our work are [BS99b, BS99a]. The algorithm is implemented in Maple, and is downloadable from `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`.

Introductory Definitions

The problem of *algorithmic resolution* of singularities, shortly, is:

Given an algebraic variety X, construct a proper, birational morphism $Y \to X$, where Y is a smooth variety.

The morphism is an isomorphism over an open dense subset of X. Along the same lines, the problem of *embedded resolution* in case of hypersurfaces is:

Given a nonsingular variety W, and a variety X of codimension one in W, construct a proper, birational morphism $\pi : W' \to W$, where W' is smooth, such that $\pi^{-1}(X)$ has at most normal crossings, and π is an isomorphism outside the singular locus of X.

Historical Review

The main milestones of the history of resolution of singularities:

- For curves, the existence of a resolution was already known in the 19th century.
- In the surface case, Walker gave an analytic proof [Wal35], and Zariski gave an algebraic proof [Zar39].

- Five years later, Zariski proved resolution of 3-folds [Zar44].
- In 1964, Hironaka came up with his famous result of resolution for arbitrary dimension [Hir64]. Hironaka's result holds only for the characteristic zero case.
- In positive characteristic, there are some partial results [Abh65, Lip78], but the general case is still open. For an account of the difficulties in positive characteristic, the reader should consult [Hau98].
- Later, [Vil89, Vil91, EV99] and [BM91, BM97] came up with constructive strategies to find (embedded) resolution. Still, the main interest of these authors was to prove the existence of resolutions with particular properties (equivariance) and not actual computation.

Recently [BS99b, BS99a] the latest, constructive results were turned into algorithms of the level of explicitness that can be executed by a computer. This conversion was not at all straightforward, and required several subproblems to cope with, like: computation of blowing ups of arbitrary nonsingular varieties efficiently, handling the factorially increasing exponents of the coefficient ideals, eliminating the redundancies of chart-coverings, finding nonsingular hypersurfaces of maximal contact, etc.

Still, there is need for speeding up the resolution process to reach wide usability, since there hasn't been found complexity bound yet for a general desingularization algorithm, treating the zero characteristic case in arbitrary dimension.

Brief Description of the Algorithm

The resolution is obtained by consecutive blowing ups of *permissible* centers in the ambient space W (see above), which has to fulfill the requirements:

- they are nonsingular and
- they have normal crossings with the exceptional divisors of the previous blowing ups.

The hypersurface is resolved through the resolution of several basic objects of the types *weighted* and *simple*. A *weighted basic object* consists of:

- W a regular variety,
- J a nonzero ideal in $k[W]$,
- b a positive integer,
- E a finite sequence of regular hypersurfaces of W with normal crossings (the list exceptional divisors of previous blowing ups).
- $a : E \to \mathbb{N}$, the weights of the exceptional divisors.

A *simple basic object*, described by (W, J, b, E) with the same meaning as above, has the property $\mathrm{ord}_x(J) = b$ for all $x \in \mathrm{Sing}(J, b)$. We say that a simple basic object is *balanced* iff for every $x \in \mathrm{Sing}(J, b)$ there exists a $z \in \Delta^{b-1}(J, b)$ of order one, where $\mathbf{V}(z)$ has normal crossings with E in a neighborhood of x. A *tagged* object is a simple object together with $E_- \subseteq E$, such that $(W, J, b, E - E_-)$ is a balanced object.

The resolution of the hypersurface $\mathbf{V}(f) \subset \mathbb{A}^n$ starts with the resolution of the weighted basic object $(\mathbb{A}^n, \langle f \rangle, 1, \text{empty\_sequence}, \text{empty\_function})$.

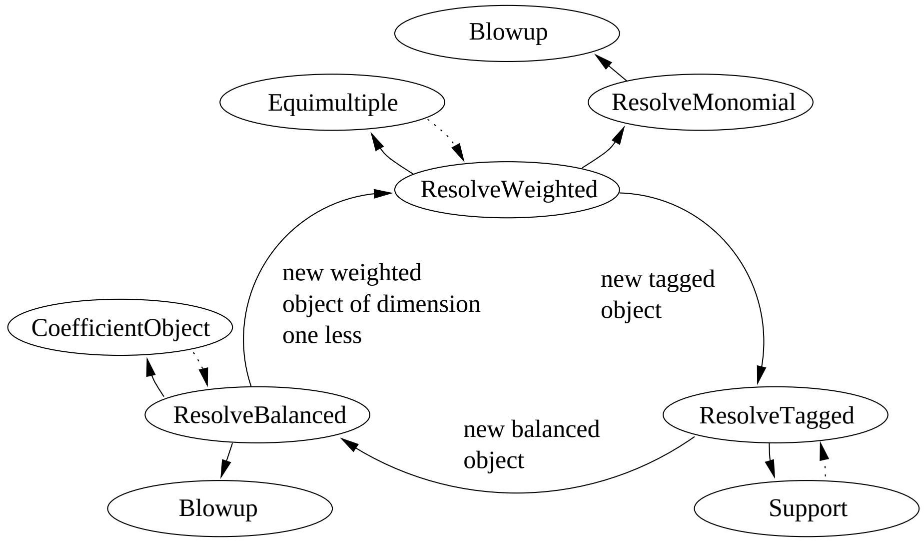

Figure 1: The functional dependencies of the subalgorithms.

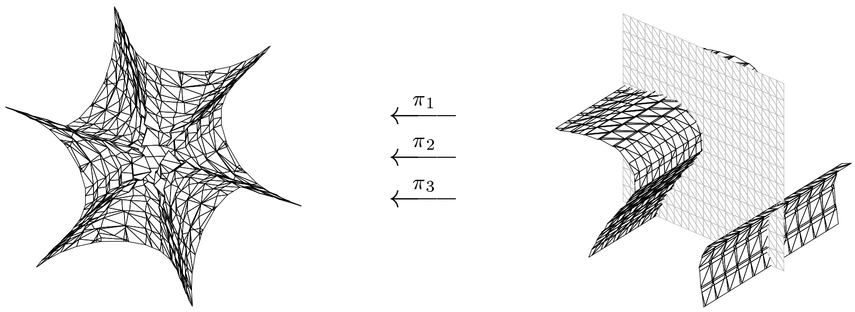

Figure 2: An embedded resolution of a singular surface in three-space.

The blowing up at the origin (see figure 2) results charts with transforms of the same structure. The π's are the morphisms of the blowing up corresponding to the charts.

The main subalgorithms of the resolution program in more detail:

The next subroutine resolves weighted basic objects, whose ideals are monomial, i.e. are of the form $\mathbf{I}(H_1)^{a_1} \cdots \mathbf{I}(H_k)^{a_k}$, where $E = \{H_1 \ldots, H_k\}$.

Algorithm *ResolveMonomial*($\mathcal{T}$);
  while not resolved do
    r := the smallest number such that r hypersurfaces in E have common singularities;
    $\overline{b}$:= the largest possible value of $a(H_1) + \ldots + a(H_r)$, where $H_1, \ldots, H_r \in E$ have common singularities;
    $H_1, \ldots, H_r$:= the first r hypersurfaces for which the maximum $\overline{b}$ is achieved (with respect to lexicographical ordering);
    Y := the intersection of $H_1, \ldots, H_r$;
    BlowUp(Y);

Algorithm *Equimultiple*(W, J, b, E, a);
  $M, \overline{J}$:= the monomial part and the proper part;
  $\overline{b}$:= the largest order of $\overline{J}$ at points in the singular locus;
  if $\overline{b} \geq b$ then
    $I := \overline{J}$;
    $w := \overline{b}$;
  else
    $I := \overline{J}^{b-\overline{b}} + M^{\overline{b}}$;
    $w := (b - \overline{b})\overline{b}$;
  $E_- := E$;
  return (W, I, w, E, E_-);

Algorithm *ResolveWeighted*($\mathcal{W}$);
  while not resolved and multiplicity> 0 do
    $\mathcal{T}$:= *Equimultiple*(current object);
    ResolveTagged($\mathcal{T}$);
  if not resolved then
    ResolveMonomial(current object);

Algorithm *ResolveBalanced*($\mathcal{B}$);
  V := a hypersurface of maximal contact;
  C := the coefficient ideal of $\mathcal{B}$ with respect to V;
  if C is the zero ideal then
    BlowUp(V);
  else
    $\mathcal{W}$:= the coefficient of object of $\mathcal{B}$ with respect to V;
    ResolveWeighted($\mathcal{W}$);

Algorithm *Support*(W, J, b, E, E_-);
    $r :=$ the largest number such that r hypersurfaces in E_- have common singularities;
    $H_1, \ldots, H_r :=$ the first r hypersurfaces in E_- having common singularities
        (with respect to lexicographical ordering);
    $I := J + \mathbf{I}(H_1)^b + \ldots + \mathbf{I}(H_r)^b$;
    $E_+ := E - E_-$;
    return (W, I, b, E_+);

Algorithm *ResolveTagged*$(\mathcal{T})$;
    while not resolved do
        $\mathcal{B} :=$ *Support*(current object);
        ResolveBalanced$(\mathcal{B})$;

On a lower level of concepts, we need to find appropriate representation for the W-varieties, so that operations like blowing up can be performed conveniently. We represent the smooth varieties by their affine covering and describe a chart (U) by the following data:

- VAR a list of affine variables,
- DEP a list of polynomials from $k[\mathrm{VAR}]$, strictly spoken the chart is $\mathbf{V}(\mathrm{DEP})$,
- IND a list of polynomials representing elements from $k[U]$, that give rise to a system of regular parameters in each point of U,
- PDER a matrix of polynomials representing elements from $k[U]$, in the ij-th position containing the partial derivative of the i-th coordinate w.r.t. the j-th parameter.

We use the three basic operations *cover*, the covering of a chart by its open affine subsets, *exchange*, the replacement of an element of IND with another element of $k[U]$, and *blowup*, the blowing up of the chart along a given permissible center.

The corresponding procedures are (assume VAR$=(x_1, \ldots, x_m)$, IND$=(P_1, \ldots, P_n)$):

Algorithm *Cover*((VAR, DEP, IND, PDER), F);
    L := ();
    for each $f \in F$ do
        VAR$'$:= append x_{m+1} to VAR;
        DEP$'$:= append $x_{m+1}f - 1$ to DEP;
        R := $(-x_{m+1}^2 \partial_k f \mid k = 1, \ldots, n)$;
        PDER$'$:= append R as the $(m+1)$-st row to PDER;
        L := attach a new chart defined by (VAR$'$, DEP$'$, IND, PDER$'$) to L;
    return L;

Where the elements of F generate the ideal $\langle 1 \rangle$.

Algorithm *Exchange*((VAR, DEP, IND, PDER), q, i, a);
    IND$'$:= replace P_i by q in IND;
    for j from 1 to m do
        PDER$'[j,i] := ap_{ji}$;
        for k from 1 to m except i do
            PDER$'[j,k] := p_{jk} - ap_{ji}\partial_k q$;
    return (VAR, DEP, IND$'$, PDER$'$);

Where q is an element from $k[U]$, i is an index to IND, such that $a = (\partial_i q)^{-1}$ exists.

Algorithm *BlowUp*((VAR, DEP, IND, PDER), $(P_1, \ldots, P_r)$);
    L := ();
    for i from 1 to r do
        VAR$'$:= append $(x_{m+j} \mid j = 1, \ldots, r,\ j \neq i)$ to VAR;
        DEP$'$:= append $(x_{m+j}P_i - P_j \mid j = 1, \ldots, r,\ j \neq i)$ to DEP;
        IND$'$:= replace P_j by x_{m+j} in IND for $j = 1, \ldots, r,\ j \neq i$;
        for j from 1 to m do
            PDER$'[j,i] := p_{ji} + \sum_{k=1,k\neq i}^{r} x_{m+k}p_{jk}$;
            for k from 1 to r except i do
                PDER$'[j,k] := P_i p_{jk}$;
            for k from $r+1$ to n do
                PDER$'[j,k] := p_{jk}$;
        for j from 1 to r except i do
            R := a list of zeros of length n with a 1 at the j-th position;
            PDER$'$:= append R as a new row to PDER$'$;
        L := append (VAR$'$, DEP$'$, IND$'$, PDER$'$) to L;
    return L;

Where the center is defined by $(P_1, \ldots, P_r)$.

Example

We use the initial variables x, y, z, and v_1, v_2 as the introduced ones. Suppose we want to resolve the surface S in three-space with equation

$$z^2 + (x^2 + y^2 - 1)^2 y = 0.$$

The circle $x^2 + y^2 - 1 = z = 0$ is a double curve of S which has to be blown up. Actually, the algorithm tells to first blow up the two points $(\pm 1, 0, 0)$. But for our purposes it suffices to consider what happens in the open chart $y \neq 0$. In this chart, the algorithm will blow up the circle.

So, we first do a cover operation, producing a chart C_1 isomorphic to the open set $y \neq 0$:

1. VAR $= [x, y, z, v_1]$,

2. DEP $= [yv_1 - 1]$,

3. $\mathrm{IND} = [x, y, z]$,

4. $\mathrm{PDER} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -v_1^2 & 0 \end{pmatrix}$.

The entry of the last row of the matrix PDER is introduced by the cover operation and comes from the formula $\partial_y v_1 = \partial_y \frac{1}{y} = -\frac{1}{y^2} = -v_1^2$. The cover operation also produces another chart C_2, isomorphic to the open sets $y^2 - 1 \neq 0$, but here we only treat C_1.

Next, we exchange y by $x^2 + y^2 - 1$ in IND. This is possible because $\partial_y(x^2 + y^2 - 1)$ is invertible in C_1. We obtain the chart E:

1. $\mathrm{VAR} = [x, y, z, v_1]$,

2. $\mathrm{DEP} = [v_1 y - 1]$,

3. $\mathrm{IND} = [x, x^2 + y^2 - 1, z]$,

4. $\mathrm{PDER} = \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2} v_1 & 0 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2} v_1^3 & 0 \end{pmatrix}$.

Now, we are ready for blowing up, and in this step we obtain two charts. The first, say, B_1 is given by

1. $\mathrm{VAR} = [x, y, v_1, v_2]$,

2. $\mathrm{DEP} = [v_1 y - 1]$,

3. $\mathrm{IND} = [x, x^2 + y^2 - 1, v_2]$,

4. $\mathrm{PDER} = \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2} v_1 & 0 \\ v_1^3 x & -\frac{1}{2} v_1^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The variable z has been eliminated. The proper transform of S has equation $v_2^2 + y$, hence it is nonsingular. The exceptional divisor is $x^2 + y^2 - 1$, and one can show easily that the total transform is a normal crossing divisor.

The second chart B_2 is given by

1. $\mathrm{VAR} = [x, y, z, v_1, v_2]$,

2. $\mathrm{DEP} = [v_1 y - 1, v_2 z - x^2 - y^2 + 1]$,

3. $\mathrm{IND} = [x, v_2, z]$,

4. $\mathrm{PDER} = \begin{pmatrix} 1 & 0 & 0 \\ -v_1x & -\frac{1}{2}v_1z & -\frac{1}{2}v_2v_1 \\ 0 & 0 & 1 \\ v_1^3x & -\frac{1}{2}v_1^3z & -\frac{1}{2}v_2v_1^3 \\ 0 & 1 & 0 \end{pmatrix}.$

Again the proper transform v_2^2y+1 is nonsingular and has normal crossing with the exceptional divisor z.

References

[Abh65] S. S. Abhyankar. Resolution of singularities of arithmetical surfaces. In *Arithmetical Algebraic Geometry*, pages 111–152. Harper and Row, 1965.

[BM91] E. Bierstone and P. Milman. A simple constructive proof of canonical resolution of singularities. In T. Mora and C. Traverso, editors, *Effective methods in algebraic geometry*, pages 11–30. Birkhäuser, 1991.

[BM97] E. Bierstone and P. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. math.*, 128:207–302, 1997.

[BS99a] G. Bodnár and J. Schicho. Automated resolution of singularities for hypersurfaces. Technical Report 99-06, RISC-Linz, Univ. Linz, A-4040 Linz, 1999.

[BS99b] G. Bodnár and J. Schicho. A computer program for the resolution of singularities. In *Working Week on Mountains and Singularities*. Obergurgl, 1999.

[EV99] S. Encinas and O. Villamayor. Constructive desingularization and equivariance. In *Working Week on Mountains and Singularities*. Obergurgl, 1999.

[Hau98] H. Hauser. Seventeen obstacles for resolution of singularities. In V. I. Arnold, G.-M. Greuel, and J. Steenbrink, editors, *Singularities*. Birkhäuser, 1998.

[Hir64] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic 0. *Ann. Math.*, 79:109–326, 1964.

[Lip78] J. Lipman. Desingularization of two-dimensional schemes. *Ann. Math.*, 107:151–207, 1978.

[Vil89] O. Villamayor. Constructiveness of Hironaka's resolution. *Ann. Scient. Ecole Norm. Sup. 4*, 22:1–32, 1989.

[Vil91] O. Villamayor. Introduction to the algorithm of resolution. In T. Mora and C. Traverso, editors, *Algebraic geometry and singularities*, pages 123–154. Birkhäuser, 1991.

[Wal35] R. J. Walker. Reduction of the singularities of an algebraic surface. *Ann. Math.*, 36:336–365, 1935.

[Zar39] O. Zariski. The reduction of singularities of an algebraic surface. *Ann. Math.*, 40:639–689, 1939.

[Zar44] O. Zariski. Reduction of the singularities of algebraic threeidimensional varieties. *Ann. Math.*, 45:472–542, 1944.