# On algorithmic desingularization of hypersurfaces

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**Abstract**

We discuss an algorithm for desingularization of hypersurfaces based on a method by Villamayor and describe its implementation.

1. The Problem
2. Applications of the Solution
3. Blowing up
4. Description of Suitable Data Types
5. Outline of the Algorithm

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# 1 What is Desingularization?

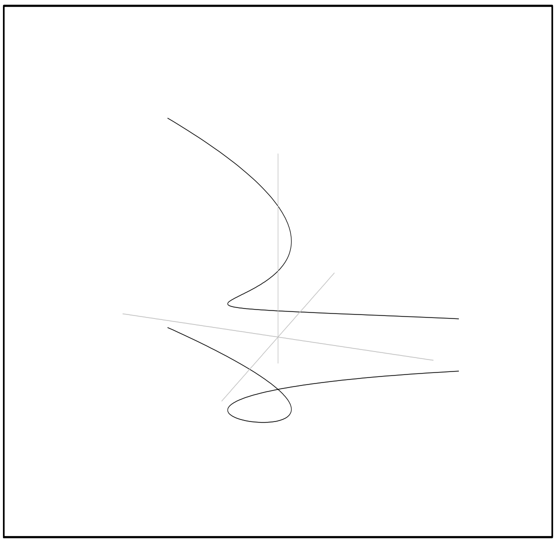
Let $X$ be an algebraic variety (over some field $k$ of characteristic zero).
A *modification* of $X$ is a variety $X'$, together with a proper birational regular map $f : X' \to X$.

**regular** means can be given by polynomials

**birational** means that there exists a partial inverse $i(f) : U \to X'$, $U \subset X$ open and dense, that can be given by rational functions

**proper** means that the pre-images of compact subsets are compact, or equivalently: $f$ is not the restriction of a proper superset $Y \supset X'$ such that $X'$ is dense in $Y$

**Example 1**

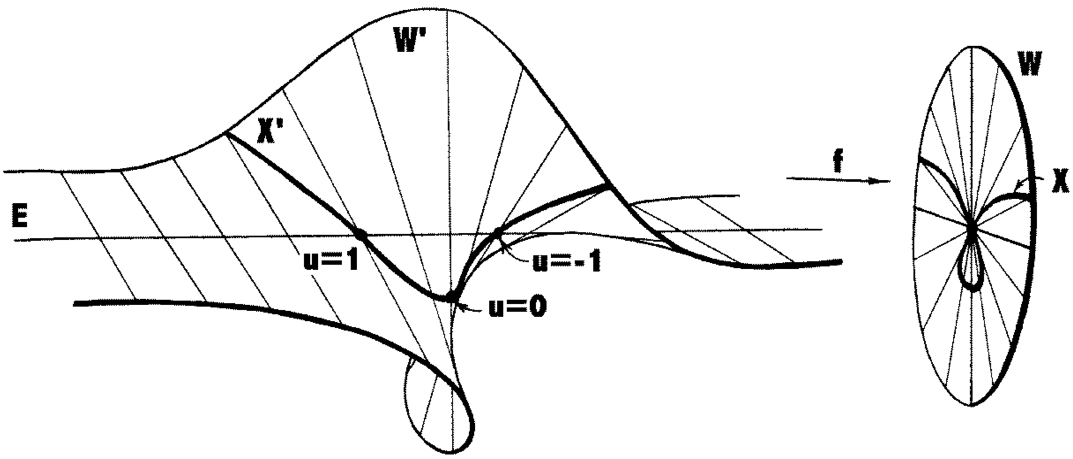


$X$ is the plane curve given by $y^2 - x^2 - x^3 = 0$.
$X'$ is the space curve $x - z^2 + 1 = y - xz = 0$.
$f : X' \to X$ is the projection $(x, y, z) \mapsto (x, y)$.
The inverse $i(f)$ is given by $(x, y) \mapsto (x, y, y/x)$.

A *desingularization* for $X$ is a modification $X'$, $f$ such that $X'$ is a nonsingular variety.

Let $W$ be a nonsingular variety and $X \subset W$ a variety of codimension 1.
If $W = \mathbf{A}^n$, then $X$ is an affine hypersurface. If $W = \mathbf{P}^n$, then $X$ is a projective hypersurface. In general, we call the pair $(X, W)$ a hypersurface (abusing language).
An *embedded desingularization* of $(X, W)$ is a modification $f : W' \to W$ of $W$, such that

- $W'$ is nonsingular,
- the the inverse image $f^{-1}(X)$ is a normal crossing divisor, i.e. a union of nonsingular hypersurfaces which intersect transversally.




An embedded desingularization of $(X, W)$ induces a desingularization of $X$. Indeed, the inverse $i(f)$ is defined on a dense subset of $X$, and the closure $X'$ of $i(f)(X)$ in $W'$ - also called the proper transform - is one of the components of the inverse image $f^{-1}(X)$. Then $(X', f|_{X'} : X' \to X)$ is a desingularization of $X$.

# 2 Applications of Desingularization

There is a huge number of theorems in algebraic geometry which depend on *existence* of desingularization/embedded desingularization. (One of the main problems in positive characteristic is that here the existence is still an open question.)
There are also many situations where one needs to *know* a desingularization.

- analysis of singularities
- computing adjoints
- local parametrization with power series
- solving differential equations near singularities

## 3 More about Adjoints

What are adjoints?

**Definition 1** *An adjoint of a hypersurface is a polynomial that vanishes with a certain pre-scribed order on the singularities of the hypersurface, including "infinitely near singularities".*

To compute a basis for the vector-space of adjoints of bounded degree, it is necessary to have a desingularization of the hypersurface.

For algebraic curves, the concept of adjoints was developed by A. Clebsch and M. Nöther in the 19th century. There exist several algorithms for computing adjoints, based on desingularization or on Puiseaux series. Using adjoints, one can accomplish various tasks, such as

- computing the genus
- if genus is zero, computing a rational parametrization (Winkler, Sendra)
- if genus is one, computing a Weierstraß normal form (van Hoeij)
- if the curve is hyperelliptic, computing a parameterization with square roots (Abhyankar)
- performing group operation on the Jacobean variety

For surfaces higher dimensional varieties, adjoints have been introduced and used by Cremona, Castelnuovo, Enriques. In the "Grothendieck area", adjoints have almost disappeared in the literature. However, it was observed that adjoints *could be used* for

- computing geometric genus, plurigenera, Kodaira-dimension
- computing arithmetic genus (Blass/Lipman)
- if $p_a = P_2 = 0$, computing a rational parametrization for surfaces (Schicho)

I guess that the computing power of adjoints in higher dimensions is even stronger than in the curve case!

# 4 History/State of the Art of the Desingularization Problem

**∼1870** desingularization of curves

**1935** desingularization of surfaces, analytic case (Walker)

**1939** desingularization of surfaces, algebraic case (Zariski)

**1944** desingularization of 3-folds (Zariski)

**1964** desingularization of $n$-folds, existence proof (Hironaka) (by the way, a non-constructive variant of Gröbner bases was invented for the proof)

**1990** desingularization of curves is polynomial (Teitelbaum)

**1991** desingularization of $n$-folds, constructive proof (Villamayor, Bierstone-Milman)

Up to now, there existed no implementation of the algorithm which is implicitly contained in the constructive proof. The reason for this is two-fold.

- the authors have primarily been interested in existence of a desingularization with particular properties, not in actual computing
- other people had too much respect for the difficulty of the desingularization problem

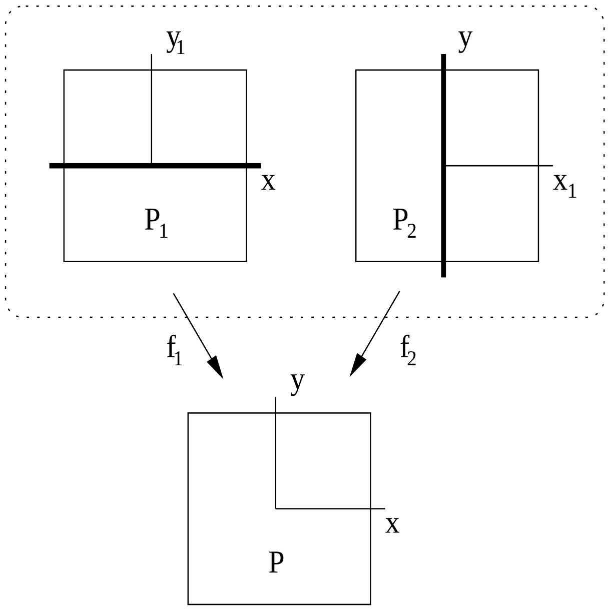
We had a two-fold reason for doing the implementation.

- we need the result for the applications
- we want to show that computing desingularizations is practically feasible with nowadays' means

# 5 Blowing up

The desingularizations by Zariski, Hironaka, Villamayor are obtained by resolution, i.e. as compositions of particular modifications called *blow-ups*: a nonsingular subvariety of codimension greater than 1 is "blown up" to a hypersurface.
The blowing-up variety comes as an abstract manifold, given by a list of charts which cover. The variety is obtained as the union of these charts modulo an equivalence relation which identifies points in the overlaps.

**Example 2** We describe the blowing up of the plane $P$ at the origin $p$.




The blowing up variety is the union of two charts $P_1, P_2$, each of them is again a plane.

The map is defined as $f_1 : P_1 \to P$, $(x, y_1) \mapsto (x, xy_1)$ in $P_1$ and as $f_2 : P_2 \to P$, $(x_1, y) \mapsto (xy_2, y)$ in $P_1$.

The inverse is defined everywhere except for $p$. If $x \neq 0$, then $i(f)(x, y) = i(f_1)(x, y) = (x, y/x) \in P_1$. If $y \neq 0$, then $i(f)(x, y) = i(f_2)(x, y) = (x/y, y) \in P_2$.

The isomorphism $i(f_2) \circ f_1 : U_1 \to U_2$, $(x, y_1) \mapsto (1/y_1, xy_1)$ identifies points on the overlaps $U_1 \subset P_1, U_2 \subset P_2$.

The inverse image of $p$ is the union of the line $x = 0$ in $P_1$ and the line $y = 0$ in $P_2$.

In general the charts of the blowing up of a nonsingular subvariety of $W$ may be different from affine space. They are, however, nonsingular varieties of the same dimension $\dim W$.

# 6 Differential Frames

If a chart is an affine space $\mathbf{A}^n$, then we are in a lucky situation:

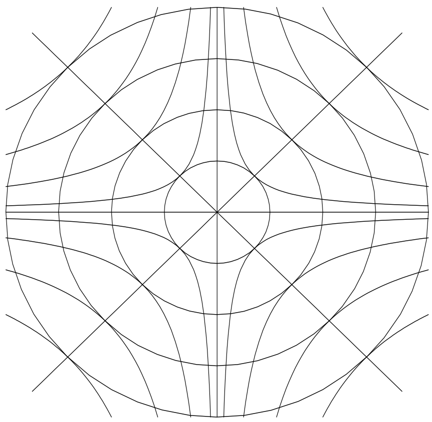
1. the partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ exist;
2. the singular set of the hypersurface $F = 0$ is the zero set of the partial derivatives of $F$;
3. the set of all points where the hypersurface has multiplicity $\geq r$ is the zero set of all partial derivatives of $F$ of order $\leq b - 1$.

In the general case, the chart $W_i$ is only a nonsingular affine variety of dimension $n$. There are usually more than $n$ affine coordinates, related by algebraic equations. The partial derivatives are not well-defined for the functions in $k[W_i]$. Computing the singular set of a hypersurface given by $F = 0$ is possible (using Jacobi determinants), but rather cumbersome. Computing multiplicities is even more cumbersome.

However, we still know that $W_i$ is nonsingular. Hence we have for every point $p \in W_i$ a local coordinate system (system of regular parameters) $(f_1, \ldots, f_n)$, which can be used for differentiation. Here $f_i$ are polynomial functions, such that their zero sets intersect transversally at $p$. Then the partial derivatives are defined locally, and the singular locus/multiplicities can be computed as in the case of affine space.

In order to define partial derivatives globally, we define: a *differential coordinate frame* for $W$ is an $n$-tuple $(f_1, \ldots, f_n)$ of functions in $k[W]$, such that for every point $p$, the translates $f_1 - f_1(p), \ldots, f_n - f_n(p)$ form a local coordinate system at $p$. With a differential frame, one can define partial derivatives which satisfy the expected properties.

**Example 3** Let $W$ be the variety $(x^2 - y^2)z = 0$ (isomorphic to the subset of $\mathbf{A}^2$ where $x^2 - y^2 \neq 0$). Then $(x^2 + y^2, xy)$ is a differential frame.



One can define the partial derivatives $\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}$, by setting

$$\begin{pmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial x}{\partial f_2} \\ \frac{\partial y}{\partial f_1} & \frac{\partial y}{\partial f_2} \end{pmatrix} = \frac{z}{2} \begin{pmatrix} x & -2y \\ -y & 2x \end{pmatrix}$$

(this is the inverse of the Jacobi-matrix), and extending to polynomials by linearity and Leibniz' rule. For instance, the partial derivatives of $F := x^2 - y^3 + 2y^2 - y$ are

$$\partial F/\partial f_1 = 2x\frac{xz}{2} + (-3y^2 + 4y - 1)(-yz),$$

$$\partial F/\partial f_2 = 2x\frac{-yz}{2} + (-3y^2 + 4y - 1)(xz),$$

and one obtains that $(0, 1, 1)$ is the only singular point of $F = 0$.

In general, a nonsingular variety might not have a differential frame, but it is always possible to cover by open charts which do have a differential frame.

# 7 Input-Output Specification

The input is an integer $n$ and a polynomial $F(x_1, \ldots, x_n)$, representing the hypersurface.
The output is a list of charts $(W, \mathrm{MAP}, \mathrm{IND}, E, P)$, where

- $W$ is an affine variety, represented by an integer $m$ and a set of polynomials in $y_1, \ldots, y_m$;
- MAP is a polynomial map, represented by $n$ polynomials in $y_1, \ldots, y_m$;
- INV is its partial inverse, represented by $m$ rational functions in $x_1, \ldots, x_n$);
- IND is a differential frame for $W$;
- $E$ is a subset of IND, representing the components of the inverse image of $ZeroSet(F)$;
- $P$ is an element of $E$ or 1, representing the proper transform. (1 in case the proper transform is disjoint from the chart.)

We do not dump the isomorphisms identifying the overlaps, because these are determined by the maps and their inverses: for any two charts, there is a birational map $\mathrm{INV}_2 \circ \mathrm{MAP}_1 : W_1 \rightarrow W_2$, which induces an isomorphism on open subsets $U_1 \subset W_1$ and $U_2 \subset W_2$. This is the identification of points in the overlap.
The list of charts defines an abstract variety, which is an embedded desingularization of $(ZeroSet(F), W)$. Note that the hypersurfaces defined by a differential frame always have normal crossing.

# 8 Idea of the Algorithm

The algorithm proceeds by a sort of double recursion on multiplicity and dimension.

- In the recursive applications of the algorithm, it is necessary not to resolve just hypersurfaces, but to resolve arbitrary ideals.
- The notion of multiplicity is defined for arbitrary ideals (generalizing multiplicity of a hypersurface).
- For given ideal $I$, we say that a hypersurface $H \subset W$ iff all functions in the ideal of $H$ are contained in the ideal generated by the derivatives of elements in $I$ of order up to $b-1$, where $b$ is the maximal multiplicity of $I$.

**Lemma 1** *Let $I$ be an ideal of $k[W]$, and let $b$ be the maximal multiplicity of $I$. Let $H$ be a nonsingular hypersurface of $W$ with maximal contact. Then there exists an ideal $C$ in $k[H]$ with the following property:*
*Any embedded resolution of $(C, H)$ induces a sequence of blowing ups of $(I, W)$, such that the multiplicity of the proper transform is less than $b$.*

Resolve(I,W):
    while (maximal multiplicity($I$) $> 1$ do
        $H$:= a nonsingular hypersurface of maximal contact;
        $C$:= an ideal as in lemma 1;
        Resolve(C,H);

# 9 How do we get a nonsingular hypersurface $H$ with maximal contact?

Locally at a $b$-fold point $p$, there exists a derivative of order $b$ which does not vanish at $p$. Thus, there exists a derivative $f$ of order $b-1$ and a coordinate function $x_n$, such that $\partial f/\partial y$ does not vanish at $p$. Then $f=0$ is the desired hypersurface.

Moreover, one can show that the functions $x_1, \ldots, x_{n-1}, f$ form a differential frame. Hence one can achieve that the hypersurface of maximal contact is a coordinate hyperplane. This has an interesting consequence: since the blowing up centers are obtained inductively by intersecting nonsingular hypersurfaces of maximal contact, these centers come as varieties given by coordinate equations of the differential frame.

Globally, nonsingular hypersurfaces with maximal contact may not exist. However, we can always cover by charts where the above construction works.

# 10 Status of the Algorithm

We are still in the "optimizing phase". The current version is yet too slow to reproduce some known resolutions of surfaces of degree 5 and 6. However, we already have some ideas from which we expect speedups, and these will be implemented in future.
Experiments showed that the practical behavior of the algorithm is quite reasonable, despite the fact that no complexity bound is known (it is difficult enough to establish termination). This is inherent to the singularity problem. On the one hand, it is very difficult to find upper bounds for the number of blowing ups required. On the other hand, it is also very difficult to find bad singularities which require many blowing ups.

The algorithm is implemented in Maple for the coefficient field of rational numbers. The only real computations are ideal membership decision (using Gröbner bases), computing representations of ideal members by a given set of generators (using extended Gröbner bases), and differentiation. Therefore, the algorithm can be adapted to arbitrary computable fields of characteristic zero (we do not even need to be able to factorize univariate polynomials).
(It is interesting to observe what happens in the case when there are isolated singularities with coordinates in some algebraic extension. In this case, the blowing up center will be the zero set of all functions in the differential frame, but this is not a single point but a set of points defined over the algebraic closure.)

The algorithm is implemented in Maple in will be included in the CASA system available at http://www.risc.uni-linz.ac.at/pub/casa.